# On the Hagedorn Behaviour of Singular Scale-Invariant Plane Waves

**Matthias Blau**

*Institut de Physique, Université de Neuchâtel*
*Rue Breguet 1, CH-2000 Neuchâtel, Switzerland*
matthias.blau@unine.ch

**Monica Borunda**

*Institut de Physique, Université de Neuchâtel*
*Rue Breguet 1, CH-2000 Neuchâtel, Switzerland*
monica.borunda@unine.ch

**Martin O'Loughlin**

*S.I.S.S.A. Scuola Internazionale Superiore di Studi Avanzati*
*Via Beirut 4, I-34014 Trieste, Italy*
loughlin@sissa.it

Abstract: As a step towards understanding the properties of string theory in time-dependent and singular spacetimes, we study the divergence of density operators for string ensembles in singular scale-invariant plane waves, i.e. those plane waves that arise as the Penrose limits of generic power-law spacetime singularities. We show that the scale invariance implies that the Hagedorn behaviour of bosonic and supersymmetric strings in these backgrounds, even with the inclusion of RR or NS fields, is the same as that of strings in flat space. This is in marked contrast to the behaviour of strings in the BFHP plane wave which exhibit quantitatively and qualitatively different thermodynamic properties.

## Contents



## 1. Introduction

The study of string theory in the cosmological setting of non-trivial time-dependent and possibly singular backgrounds is an outstanding problem. While string theory in general time-dependent backgrounds is hard, in principle string theory in plane waves (or, say, flat orbifolds) is exactly solvable. Hence one can hope to gain some insight into the nature and properties of string theory in general time-dependent backgrounds by studying it in the context of time-dependent plane waves (or time-dependent orbifolds). With this motivation, in [1, 2, 3, 4] we set out to systematically identify tractable yet physically interesting time-dependent plane wave metrics and develop tools for the quantisation of string theory in such backgrounds.[1]

In [3, 4] it was found that the plane wave metric

$$ds^2 = 2dudv + A_{ab}(u)x^a x^b du^2 + d\vec{x}^2 \tag{1.1}$$

associated to every space-time metric and choice of null geodesic in that space-time by means of the Penrose limit construction [6, 7, 8], encodes covariant information about the rate of growth of curvature and geodesic deviation along the null geodesic. In particular, singularities of $A_{ab}(u)$ result from curvature singularities of the original space-time.

Moreover, in [3, 4] we studied the Penrose limits of metrics with space-time singularities. We found that the resulting plane wave metrics for a large class of black hole,

[1] For a detailed review of recent work on time-dependent orbifolds see [5].

cosmological and null singularities (namely all metrics with singularities of power-law type [9, 10, 11] subject to some energy condition) exhibit a remarkably universal leading $u^{-2}$-behaviour,

$$A_{ab}(u) \sim u^{-2} \quad , \tag{1.2}$$

near the singularity. Plane waves with such a wave profile are scale invariant, i.e. invariant under the scaling (boost) $(u, v) \to (\lambda u, \lambda^{-1} v)$ and hence also homogeneous [8, 12, 1]. This scale invariance will turn out to play an important role in the following.

Thus the considerations of [3, 4] single out the singular scale-invariant plane waves with profile $\sim u^{-2}$ as the backgrounds to consider in order to obtain insight into the properties of string theory near physically reasonable space-time singularities. String theory in these singular homogeneous plane wave backgrounds is exactly solvable [13, 12, 2], and various aspects of string theory in this class of backgrounds have already been studied in particular in [12].

Recently, motivated by the BMN correspondence [14], thermodynamical aspects of string theory in the BFHP plane wave [15, 7, 16, 17] and other plane waves with constant (i.e. $u$-independent) profile $A_{ab}$, have been investigated in e.g. [18]-[25]. In particular, it was found that the Hagedorn behaviour of strings in such backgrounds (and with RR flux) differs quantitatively and qualitatively from that of strings in flat space.

Here we wish to study the analogous question for strings in the singular scale-invariant plane wave backgrounds with profile (1.2). In order to address this issue we need to first come to terms with the fact that string theory in these time-dependent singular homogeneous plane waves leads to a time-dependent light-cone Hamiltonian. Thus the study of the "thermodynamics" of such systems requires some care.

Following [26, 27, 28], one can study the evolution of non-equilibrium systems with respect to invariants of the system. Given such an invariant $I$, i.e. an operator which satisfies $dI/dt = 0$, one can introduce an analogue of the Boltzmann thermal state, the density operator

$$\hat{\rho}_I = \frac{\mathrm{e}^{-\beta I}}{\mathrm{tr}\,\mathrm{e}^{-\beta I}} \quad . \tag{1.3}$$

This density operator satisfies the quantum counterpart of the classical Liouville theorem for the phase space density, namely $\mathrm{tr}\,\hat{\rho}_I = 1$ and $d\hat{\rho}_I/dt = 0$. A convenient choice of invariant for a system with a time-dependent Hamiltonian $H(t)$ is $I = H(t_0)$, as it reduces to the standard choice in the case of a time-independent system and provides an "adiabatic" approximation to the system provided that $H(t)$ varies sufficiently slowly with time near $t = t_0$.

We will study the corresponding "thermal" partition function

$$Z_I(\beta) = \mathrm{tr}\,\mathrm{e}^{-\beta I} \tag{1.4}$$

for strings in the singular scale-invariant plane waves. Our main result is that the partition fuction for bosonic or type II strings, even with the inclusion of RR fields, diverges at a critical value $\beta_c$ of the parameter $\beta$ which is identical to the inverse Hagedorn temperature $\beta_c = 1/T_H$ of strings in flat space [29]. This is in marked contrast to the thermodynamical

behaviour of strings in the non-singular constant $A_{ab}$ plane waves which, as mentioned above, can be quite different.

The calculation establishing this result highlights the significance and implications of the scale invariance of these plane waves. Indeed scale invariance can be seen to imply that the string mode frequencies are independent of the light-cone momentum and hence uniformly approach the flat space frequencies at large string mode number $n$.

This may be an indication that string propagation in these scale invariant plane waves, which arise as Penrose limits of space-time singularities, has properties more in common with the propagation of strings in flat space than either have with strings in the BFHP and other time-independent plane waves.

In section 2 we summarise the approach of [26, 27, 28], to non-equilibrium thermodynamics via invariants. In section 3, following [12] we present the light-cone Hamiltonian of type IIB superstring theory in singular homogeneous plane wave backgrounds and exhibit its relation to our preferred invariant. The implications of the scale invariance are explored in section 4 and the detailed computation of the partition function is given in section 5.

## 2. Invariants and Thermodynamics of Time-Dependent Systems

For present purposes we will adopt the point of view (see e.g. [26, 27, 28]) that suitable analogues of the Boltzmann thermal state for a time-independent Hamiltonian system,

$$\hat{\rho}_H = \frac{\mathrm{e}^{-\beta H}}{\mathrm{tr}\,\mathrm{e}^{-\beta H}} \ , \tag{2.1}$$

can be constructed as density operators of the form

$$\hat{\rho}_I = \frac{\mathrm{e}^{-\beta I}}{\mathrm{tr}\,\mathrm{e}^{-\beta I}} \ , \tag{2.2}$$

where $I$ is an *invariant* of the system, i.e. a possibly explicitly time-dependent operator in the Heisenberg picture satisfying the equation

$$\frac{d}{dt}I = \frac{\partial}{\partial t}I + i[H,I] = 0 \ . \tag{2.3}$$

Note that for a time-independent system evidently the Hamiltonian $H$ itself satisfies this equation. In particular, the density operator satisfies the quantum counterpart of the classical Liouville theorem for the phase space density, namely $\mathrm{tr}\,\hat{\rho}_I = 1$ and the Liouville-von Neumann equation

$$\frac{d}{dt}\hat{\rho}_I = \frac{\partial}{\partial t}\hat{\rho}_I + i[H,\hat{\rho}_I] = 0 \ . \tag{2.4}$$

As a consequence of this equation, the density operator $\hat{\rho}_I(t)$ allows one to calculate the time evolution of the expectation value of any operator $\mathcal{O}$ in the mixed state characterised by the invariant $I$,

$$< \mathcal{O} >_I (t) = \mathrm{tr}(\mathcal{O}\hat{\rho}_I) \ . \tag{2.5}$$

Different choices of $I$ correspond to different initial "thermal" ensembles with partition functions

$$Z_I(\beta) = \mathrm{tr}\, \mathrm{e}^{-\beta I} \quad . \tag{2.6}$$

In particular, in [26] it has been shown that this reproduces and unifies various approaches to non-equlibirum thermodynamics such as mean-field or Hartree-Fock methods.

The parameter $\beta$ is in many ways analogous to an inverse temperature $1/T$. However, here and in the following we occasionally put the word "thermal" in quotes to emphasise that we are not claiming that (2.2) describes "the system at temperature $\beta = 1/T$".[2]

For the harmonic oscillator systems under consideration (namely the light-cone Hamiltonians for strings on time-dependent plane waves), invariants are easy to come by, e.g. using invariant oscillators (Appendix A).[3]

A convenient, but by no means the unique acceptable, choice is the "instantaneous" density operator

$$I = H(t_0) \tag{2.7}$$

(which will be explicitly time-dependent when written in terms of Heisenberg operators). The corresponding density operator reduces to the standard choice in the case of a time-independent system, and it provides an "adiabatic" approximation to the system provided that $H(t)$ varies sufficiently slowly with time near $t = t_0$.

## 3. The Light-Cone Hamiltonian and the Invariant

We now summarise some results from [12] regarding the light-cone Hamiltonian for bosonic strings in a purely dilatonic singular homogeneous plane wave background (we will briefly discuss other string theories and backgrounds at the end of section 5).

In the light cone gauge

$$U(\sigma, t) = 2\alpha' p_v t \quad , \tag{3.1}$$

($p_v = p^u$ the light-cone momentum) the dynamics of the transverse string coordinates $X^a(\sigma, t)$ in the plane wave metric

$$ds^2 = 2dudv + A_{ab}(u)x^a x^b du^2 + d\vec{x}^2 \quad , \tag{3.2}$$

is governed by the quadratic light-cone Hamiltonian

$$H_{lc}(t) = -p_u = \frac{1}{8\pi\alpha'^2 p_v} \int_0^\pi d\sigma \, (\dot{X}^{a2} + X'^{a2} - 4\alpha'^2 p_v^2 A_{ab}(2\alpha' p_v t) X^a X^b) \quad . \tag{3.3}$$

Note that precisely for the singular plane waves with profile

$$A_{ab}(u) = -\omega_a^2 \delta_{ab} u^{-2} \tag{3.4}$$

[2] Such an identification has been used in [27] to define an analogue of temperature for non-equilibrium systems.

[3] See [1] for a derivation of the Lewis-Riesenfeld theory of invariants of time-dependent harmonic oscillators [30] from the geometry of plane waves.

the dependence of the light-cone Hamiltonian on $p_v$ disappears (up to an overall factor). This is a consequence of the *scale invariance*

$$(u,v) \to (\lambda u, \lambda^{-1} v) \tag{3.5}$$

characterising the corresponding homogeneous plane wave metric [1].

Thus the Fourier modes of the string are harmonic oscillators with frequencies (up to a standard overall factor of $1/\alpha' p_v$)

$$\omega_n^a(t) = \sqrt{n^2 + \frac{\omega_a^2}{4t^2}} \ . \tag{3.6}$$

We will consider the case that all the frequencies $\omega_a$ are real. This excludes vacuum plane waves, but includes e.g. the Penrose limits of FRW metrics [8, 3] and a large class of other power-law singularities [4].[4]

The natural mode expansion in terms of solutions to the classical equations of motion and the corresponding invariant oscillators $\alpha_n^a$ (Appendix A) leads to a non-diagonal light-cone Hamiltonian operator, namely (cf. [12] and (A.7))

$$H_{lc}(t) = \frac{1}{\alpha' p_v}[H_0(t) + \frac{1}{2}\sum_{n=1}^{\infty}\sum_{a=1}^{d}[\Omega_n^a(t)(\alpha_{-n}^a\alpha_n^a + \tilde{\alpha}_{-n}^a\tilde{\alpha}_n^a) - B_n^a(t)\alpha_n^a\tilde{\alpha}_n^a - B_n^{a\star}(t)\alpha_{-n}^a\tilde{\alpha}_{-n}^a]] \tag{3.7}$$

where $H_0(t)$ is the Hamiltonian of the zero modes and the time-dependent coefficients are quadratic expressions in the Bessel function solutions to the string mode equations.

As shown in this specific example in [12], and discussed more generally in Appendix A in a quantum mechanical context, the light-cone Hamiltonian operator simplifies significantly when written in an explicitly time-dependent but diagonalising basis of oscillators $a_n^a(t), \tilde{a}_n^a(t)$, in which the coefficients are just the appropriately normalised classical frequencies (3.6) rather than the complicated functions $\Omega_n^a(t)$ appearing in (3.7),

$$H_{lc}(t) = \frac{1}{\alpha' p_v}[H_0(t) + \sum_{n=1}^{\infty}\sum_{a=1}^{d}\frac{\omega_n^a(t)}{n}(a_n^{a\dagger}(t)a_n^a(t) + \tilde{a}_n^{a\dagger}(t)\tilde{a}_n^a(t))] + h(t) \ . \tag{3.8}$$

Here $h(t)$ is a normal ordering c-function,

$$h(t) = \sum_{a=1}^{d}(\sum_{n=1}^{\infty}\sqrt{n^2 + \frac{{\omega_a}^2}{4t^2}} + \frac{\omega_a}{4t}) \ , \tag{3.9}$$

which we will discuss in more detail in section 5. The oscillators are normalised (as are the $\alpha$ oscillators in (3.7)) to have the non-zero commutation relations

$$\begin{aligned} [a_n^a(t), a_m^{b\dagger}(t)] &= n\delta_{n+m}\delta^{ab} \\ [\tilde{a}_n^a(t), \tilde{a}_m^{b\dagger}(t)] &= n\delta_{n+m}\delta^{ab} \ . \end{aligned} \tag{3.10}$$

In particular, in this basis the invariant

$$I = H_{lc}(t_0) \tag{3.11}$$

takes a very simple form and its corresponding "thermal" partition function is easy to calculate (see section 5).

[4]For a dicussion of some aspects of "imaginary" frequencies in plane waves see e.g. [31, 32].

## 4. Implications of Scale Invariance

As we have seen, the string mode frequencies in the scale invariant plane wave are the $p_v$-independent but $t$-dependent

$$\omega_n^a(t) = \sqrt{n^2 + \frac{\omega_a^2}{4t^2}} \quad . \tag{4.1}$$

This highlights the special feature of the scale invariant plane waves, namely that for fixed $t$ their large $n$ behaviour is exactly that of flat space,

$$\omega_n^a \stackrel{n\to\infty}{\longrightarrow} n \quad . \tag{4.2}$$

The above behaviour of strings in scale invariant homogeneous plane waves should be contrasted with the behaviour of the string modes in the plane waves with constant wave profile

$$A_{ab}(u) = -\mu_a^2 \delta_{ab} \quad . \tag{4.3}$$

In this case one finds (up to the same overall factor as in (3.6)) the now $t$-independent but $p_v$-dependent frequencies [16, 17]

$$\tilde{\omega}_n^a(p_v) = \sqrt{n^2 + \alpha'^2 p_v^2 \mu_a^2} \quad . \tag{4.4}$$

While these also behave as $\sim n$ for large $n$ and fixed $p_v$, the integration over the light-cone momentum that arises in the calculation of the partition function implies that for the symmetric (i.e. constant $A_{ab}$) plane waves, or any $u$-dependence of the frequencies other than $u^{-2}$ times a bounded function of $u$, there can be significant departures from the flat space behaviour even for $n \to \infty$.

This seems to indicate that the scale invariant singular homogeneous plane waves have properties more in common with strings in flat space than either have with the symmetric plane waves. Another manifestation of this is the fact that for $u > 0$ the $\omega_a \to 0$ limit of the metric with profile (3.4) is the flat metric, while rescaling $\mu_a$ in (4.3) is an isometry of the metric (so that the flat space emerging at $\mu_a = 0$ is a non-Hausdorff limit).

In particular, this has direct implications for the issue we want to study in this paper, namely the Hagedorn behaviour of strings in scale invariant plane waves, which depends on the properties of the string modes and the exponential growth of the number of states at large $n$. On the basis of the above reasoning, one might expect this behaviour to be identical to that in flat space, and we will confirm this by an explicit calculation in the next section. On the other hand, as was already mentioned in the Introduction, the symmetric plane waves exhibit a different behaviour.

## 5. The "Thermal" Partition Function

In this section we will present the calculation of the thermal partition function (2.6) corresponding to the invariant $I = H_{lc}(t_0)$ (3.11). We will add to the invariant the light-cone momentum $p_v$ in such a way that in the flat space limit the density operator reduces to the standard expression for the time-like Hamiltonian $(p_u + p_v)/\sqrt{2}$.

Thus the density operator of interest is

$$\hat{\rho}_I = \frac{e^{-\frac{\beta}{\sqrt{2}}(I+\hat{p}_v)}}{\mathrm{tr}\, e^{-\frac{\beta}{\sqrt{2}}(I+\hat{p}_v)}} \quad , \tag{5.1}$$

and we will now study the associated thermal partition function

$$Z_I(\beta) = \mathrm{tr}\, e^{-\frac{\beta}{\sqrt{2}}(\hat{p}_v + I)} \quad . \tag{5.2}$$

To evaluate this trace, we need to implement the level matching condition. We first do the calculation for a plane wave solution with non-trivial dilaton and no other non-trivial fields in the bosonic string theory [12] and will discuss later the generalisation to other string theories and field configurations.

In terms of either of the oscillator bases the level matching condition is simply the difference between the left and right number operators,

$$N - \tilde{N} = \sum_{a=1}^{d}\sum_{n=1}^{\infty}(\alpha_{-n}^a \alpha_n^a - \tilde{\alpha}_{-n}^a \tilde{\alpha}_n^a) = \sum_{a=1}^{d}\sum_{n=1}^{\infty}(a_n^{a\dagger} a_n^a - \tilde{a}_n^{a\dagger}\tilde{a}_n^a) \quad . \tag{5.3}$$

Thus $Z_I(\beta)$ is

$$Z_I(\beta) = \int dp_v \int d\lambda\, e^{-\frac{\beta p_v}{\sqrt{2}}}\ \mathrm{tr}\, e^{-\frac{\beta}{\sqrt{2}}I + 2\pi i\lambda(N-\tilde{N})} \quad . \tag{5.4}$$

For notational simplicity we will from now on consider the case where all the frequencies are equal, $\omega_a = \omega$, but our final result for the critical (Hagedorn) value of $\beta$ turns out to be independent of $\omega$ and is hence, in particular, also valid when the frequencies are distinct. We will also abbreviate $\omega_n(t_0) \equiv \omega_n$.

In terms of the complex variable $\tau$,

$$\tau = \tau_1 + i\tau_2 = \lambda + i\frac{\beta}{2\sqrt{2}\alpha'\pi p_v} \quad , \tag{5.5}$$

the trace inside the integral is

$$\begin{aligned}
\mathrm{tr}\, e^{-2\pi\tau_2 I + 2\pi i\tau_1(N-\tilde{N})} &= \prod_{n=1}^{\infty} |\sum_{m=0}^{\infty} e^{(-2\pi\tau_2\omega_n + 2\pi i\tau_1 n)m}|^{2d} \\
&\quad \times \sum_{m=0}^{\infty} e^{-2\pi\tau_2\omega_0 dm} e^{-2\pi\tau_2 h(t_0)} \\
&= (\prod_{n=-\infty}^{\infty} \frac{1}{1 - e^{-2\pi\tau_2\omega_n + 2\pi i\tau_1 n}})^d e^{-2\pi\tau_2 h(t_0)} \\
&= D_{0,0}(\tau_1, \tau_2; \frac{\omega}{2t_0})^{-d}\, e^{2\pi\tau_2(d\Delta_0(q) - h(t_0))}
\end{aligned} \tag{5.6}$$

where $D_{0,0}(\tau; q)$ is a generalised (massive) theta function (see Appendix B for the definition).

The normal ordering term

$$h(t_0) = d(\sum_{n=1}^{\infty} \sqrt{n^2 + \frac{\omega^2}{4t_0^2}} + \frac{\omega}{4t_0}) \ , \tag{5.7}$$

is divergent. Actually $h(t_0)$ has two divergences, the first one arising from a sum over $n$ of $n$ - a quadratic divergence, the second one arising from a sum over $n$ of $1/n$ - a logarithmic divergence. In the present case (generalising bosonic string theory in Minkowski background) this quadratic divergence is cancelled by a generalised zeta function regularisation. The details of the resummation are given in Appendix B with the result that ($q = \omega/2t_0$),

$$\frac{h(t_0)}{d} = \frac{1}{2} \sum_{n \in Z} \sqrt{n^2 + q^2} \tag{5.8}$$

$$= -\frac{q}{\pi} \sum_{k=1}^{\infty} \frac{K_1(2\pi k q)}{k} - \frac{q^2}{4} \Gamma(-1) \tag{5.9}$$

$$\equiv \Delta_0(q) - \frac{q^2}{4} \Gamma(-1) \ , \tag{5.10}$$

where $K_1$ is a modified Bessel function of the second kind.

For $q = 0$ (Minkowski space) it is easy to see that we get the usual result of zeta function regularisation that $h(t_0) = -d/12$ (see Appendix B). However, for $q \neq 0$ the term proportional to $q^2$ needs interpreting as it is infinite. The source of this infinity is the subleading logarithmic divergence $\sim \zeta(1)$ of $h(t_0)$ mentioned above, as can also be seen directly by expanding and resumming the series for small $q$ by utilising the binomial expansion of the square root (B.9).

This remaining divergence is more subtle, but before we consider it in more detail we first make the following observation. Since, in the present context, $q$ is not related to the modular parameter, i.e. independent of $p_v$ because of scale invariance, the relevant modular transformation rule of $D_{0,0}(\tau; q)$ is [5]

$$D_{0,0}(\tau_1, \tau_2; q) = D_{0,0}(-\tau_1/|\tau|^2, \tau_2/|\tau|^2; q|\tau|) \ . \tag{5.11}$$

As a consequence $\tau_2 q^2$ is itself modular invariant,

$$\tau_2 q^2 = (\tau_2/|\tau|^2)(q|\tau|)^2 \ , \tag{5.12}$$

and thus any term in $h(t_0)$ that is proportional to $q^2$ makes a contribution to (5.6) that is modular invariant all by itself.

In principle there can be a logarithmic divergence arising in a two-dimensional non-linear sigma model though it generally vanishes on shell or can be removed by a singular field redefinition as such models are formally (1-loop) scale invariant. In string theory one requires that the quantisation preserves modular invariance and divergences are again regularised via singular field redefinitions and appropriate subtractions (for a discussion of

[5] This is different from the modular transformation for the BFHP plane wave, for which $q$ depends on $p_v$ (4.4) and one has $D_{0,0}(\tau; q) = D_{0,0}(-1/\tau; q/|\tau|)$ instead, cf. the discussion in [18].

these issues in the present context, see [12]). For us at the moment it is important simply to note that any subtraction that removes the divergence will be proportional to $q^2$ and thus will not destroy the modular covariance of the state sum.

Putting everything together, and setting $d = 24$ for the bosonic string, we see that we can now write $Z_I(\beta)$ as

$$Z_I(\beta) = \int_0^\infty \frac{d\tau_2}{\tau_2^2} \int_{-1/2}^{1/2} d\tau_1 \, \mathrm{e}^{-\frac{\beta^2}{4\pi\alpha'\tau_2}} D_{0,0}(\tau_1, \tau_2; \frac{\omega}{2t_0})^{-24} . \tag{5.13}$$

The potential divergence in this integral arises from the region where $p_v \to \infty$, corresponding to $\tau_2 \to 0$, and we can set $\tau_1 = 0$ to determine the leading behaviour of the integral.

To easily determine the behaviour of the integrand one can use the modular transformation property (5.11) of $D_{0,0}$ to deduce that for $\tau_1 = 0$ and in the limit $\tau_2 \to 0$ the transverse partition function behaves as

$$D_{0,0}(\tau_1, \tau_2; \frac{\omega}{2t_0})^{-24} \to \exp[-\frac{2\pi}{\tau_2} 24\Delta_0(\frac{\omega}{2t_0}\tau_2)] \ . \tag{5.14}$$

Combining this with the expansion of $\Delta_0$ (Appendix B),

$$\Delta_0(\frac{\omega}{2t_0}\tau_2) = -\frac{1}{12} + \frac{1}{2}\frac{\omega}{2t_0}\tau_2 \ , \tag{5.15}$$

one finds that the leading behaviour of the integrand as $\tau_2 \to 0$ is

$$\exp[\frac{1}{4\pi\alpha'\tau_2}(16\pi^2\alpha' - \beta^2)] \ .$$

Thus $Z_I(\beta)$ diverges for $\beta \leq \beta_c$, with

$$\beta_c^2 = 16\pi^2\alpha'. \tag{5.16}$$

This is precisely the value $\beta_c = \beta_H$ that corresponds to the Hagedorn temperature for strings in flat space [29]. Note that, in particular, this result is independent of $\omega$ or, more generally, of the frequencies $\omega_a$ determining the plane wave background. It is also manifestly independent of $t_0$ or, in other words, independent of the choice of invariant $I = H(t_0)$ determining the thermal ensemble.

For the type II superstring theories, we have a similar dilatonic background. The bosonic contribution to the "thermal" partition function $Z_I(\beta)$ is as above, with $d = 8$. To include the fermionic contribution we note that in the light cone gauge the fermionic Lagrangian is given by

$$L_F = \psi^i \partial_{\bar{z}} \psi^i + \bar{\psi}^i \partial_z \bar{\psi}^i \,, \tag{5.17}$$

so the fermions do not couple to the parameters of the metric, and therefore its contribution to $Z_I(\beta)$ will be the same as in a flat background. It follows that $Z_I(\beta)$ diverges for $\beta \leq \beta_c$ where $\beta_c = \beta_H = 2\pi\sqrt{2\alpha'}$ coincides with the inverse Hagedorn temperature of type II strings in flat space [29].

For superstring backgrounds that also have non-vanishing form fields one finds the same results. For example, consider a type IIB superstring propagating in an homogeneous plane wave metric supported by a Ramond-Ramond 5-form background (the singular analogue [12, 1] of the BFHP background)

$$\begin{aligned} ds^2 &= 2dudv - \frac{\omega^2}{u^2}\vec{x}^2du^2 + d\vec{x}^2 \\ F_{u1234} &= F_{u5678} = 2\frac{\omega}{u} \quad . \end{aligned} \tag{5.18}$$

As in [16, 17], the GS light-cone gauge action for a string in this background is

$$\begin{aligned} L &= L_B + L_F \\ L_B &= \frac{1}{2}(\partial_+ X^a \partial_- X^a - \frac{\omega^2}{t^2}X^{a2}) \\ L_F &= i(\theta^1\bar{\gamma}^-\partial_+\theta^1 + \theta^2\bar{\gamma}^-\partial_-\theta^2 - 2\frac{\omega}{t}\theta^1\bar{\gamma}^-\Pi\theta^2). \end{aligned} \tag{5.19}$$

Notice that once again all $p_v$-dependence has disappeared. The mode equations for the fermions can also be solved in closed form (in terms of Bessel functions). Just as for the bosonic modes, their large $n$ behaviour uniformly aproaches that of flat space. Thus the resulting critical "inverse temperature" $\beta_c$ is independent of $\omega$ and equal to the type II Hagedorn temperature of flat space, in contrast to the result for the BFHP (and other symmetric) plane waves [18]-[22]. For other (e.g. NS) backgrounds, the string mode equations may be more complicated (and difficult to solve in closed form), but the large $n$ behaviour will always be as above, leading to the same conclusions regarding the value of $\beta_c$.

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## A. Invariant vs. Diagonalising Oscillator Bases

Consider the time-dependent harmonic oscillator with Hamiltonian

$$H = \frac{1}{2m}p^2 + \frac{m\omega(t)^2}{2}x^2 \quad . \tag{A.1}$$

For a time-*in*dependent harmonic oscillator one can easily find an oscillator basis which gives a *diagonal* Hamiltonian and such that the oscillators themselves are *invariant*. For a time-dependent oscillator one cannot find a basis for which these two conditions are simultaneously satisfied.

It is easy to find invariant oscillators: one defines $\alpha$ and $\alpha^\dagger$ by the "mode expansion"

$$\hat{x}(t) = \alpha z(t) + \alpha^\dagger z(t)^* \tag{A.2}$$

$$\hat{p}(t) = m(\alpha\dot{z}(t) + \alpha^\dagger\dot{z}(t)^*) \tag{A.3}$$

where $z(t)$ is a complex solution to the equations of motion

$$\ddot{z}(t) = -\omega(t)^2 z(t) \tag{A.4}$$

with the Wronskian of $z(t)$ normalised to

$$W(z, z^*) = z(t)\dot{z}^*(t) - z(t)^*\dot{z}(t) = \frac{i\hbar}{m} \quad . \tag{A.5}$$

The Heisenberg operator equations of motion imply that indeed $\alpha$ is invariant, in the sense that it satisfies (2.3), and the Wronskian normalisation condition implies that these oscillators satisfy the standard canonical commutation relations

$$[\alpha, \alpha^\dagger] = 1 \quad . \tag{A.6}$$

Note that then also any, say, quadratic function of these oscillators with time-independent coefficients is an invariant in the sense of (2.3).

In this basis the Hamiltonian takes the general form

$$\hat{H} = \frac{m}{2}[(\alpha\alpha^\dagger + \alpha^\dagger\alpha)(|\dot{z}|^2 + \omega^2|z|^2) + \alpha^2(\dot{z}^2 + \omega^2 z^2) + \alpha^{\dagger 2}(\dot{z}^{*2} + \omega^2 z^{*2})] \quad . \tag{A.7}$$

It is non-diagonal for $\omega(t)$ not constant, because

$$\frac{d}{dt}(\dot{z}(t)^2 + \omega(t)^2 z(t)^2) = 2\omega(t)\dot{\omega}(t)z(t)^2 \tag{A.8}$$

is then not zero, so a fortiori $\dot{z}^2 + \omega^2 z^2$ itself cannot be zero.

For $\omega(t) = \omega$ constant, the Hamiltonian is diagonal and explicitly the above mode expansion reads

$$\begin{aligned} \omega(t) = \omega \Rightarrow \quad \hat{x}(t) &= \sqrt{\frac{\hbar}{2m\omega}}(\alpha \mathrm{e}^{-i\omega t} + \alpha^\dagger \mathrm{e}^{i\omega t}) \\ \hat{p}(t) &= i\sqrt{\frac{\hbar m\omega}{2}}(-\alpha \mathrm{e}^{-i\omega t} + \alpha^\dagger \mathrm{e}^{i\omega t}) \quad . \end{aligned} \tag{A.9}$$

Since the proof that the Hamiltonian in this basis is diagonal is purely algebraic, i.e. does not depend on the $t$-independence of $\omega$, we are thus led to define, in general, an alternative oscillator basis $a$ and $a^\dagger$ by

$$\hat{x}(t) = \sqrt{\frac{\hbar}{2m\omega(t)}}(a\mathrm{e}^{-i\omega(t)t} + a^\dagger \mathrm{e}^{i\omega(t)t}) \tag{A.10}$$

$$\hat{p}(t) = i\sqrt{\frac{\hbar m\omega(t)}{2}}(-a\mathrm{e}^{-i\omega(t)t} + a^\dagger \mathrm{e}^{i\omega(t)t}) \quad . \tag{A.11}$$

This indeed defines the oscilators $a$ and $a^\dagger$, as one can solve for them in terms of $\hat{x}(t)$ and $\hat{p}(t)$, and this determines the non-trivial time-dependence of $a(t)$ and $a^\dagger(t)$. Nevertheless one has the time-independent canonical commutation relations

$$[a(t), a^\dagger(t)] = 1 \quad . \tag{A.12}$$

In terms of these oscillators the Hamiltonian is diagonal,

$$\hat{H}(t) = \frac{\hbar\omega(t)}{2}(aa^\dagger + a^\dagger a) \tag{A.13}$$

and, in particular, the frequency in the diagonal basis is just the classical frequenncy $\omega(t)$.

Explicitly, the time-dependent $SU(1,1)$ transformation ( = invariance group of the oscillator algebra) from $\alpha$ to $a$ is

$$a = f(t)\alpha + g(t)\alpha^\dagger \quad , \tag{A.14}$$

where

$$\begin{aligned} f(t) &= \sqrt{\frac{m}{2\hbar\omega(t)}}\mathrm{e}^{i\omega(t)t}(\omega(t)z(t) + i\dot{z}(t)) \\ g(t) &= \sqrt{\frac{m}{2\hbar\omega(t)}}\mathrm{e}^{i\omega(t)t}(\omega(t)z(t)^* + i\dot{z}(t)^*) \quad , \end{aligned} \tag{A.15}$$

satisfy the $SU(1,1)$ condition

$$|f(t)|^2 - |g(t)|^2 = -\frac{mi}{\hbar}W = 1 \tag{A.16}$$

as a consequence of the condition on the Wronskian of $z(t)$.

## B. Generalised Theta Functions

The generalised Theta function is given by [33]

$$D_{b_1,b_2}(\tau_1,\tau_2;q) \equiv \mathrm{e}^{2\pi\tau_2\Delta_{b_1}(q)} \prod_{n=-\infty}^{\infty} (1 - \mathrm{e}^{2\pi\tau_2\sqrt{(n+b_1)^2+q^2} + 2\pi i\tau_1(n+b_1) - 2\pi i b_2}) , \tag{B.1}$$

where

$$\Delta_b(q) \equiv -\frac{q}{\pi}\sum_{p=1}^{\infty}\frac{\cos(2\pi bp)}{p}K_1(2\pi qp) , \tag{B.2}$$

and $K_1$ is a modified Bessel function of the second kind. Its modular properties are

$$D_{b_1,b_2}(\tau_1,\tau_2;q) = D_{b_2,-b_1}(-\frac{\tau_1}{|\tau|^2},\frac{\tau_2}{|\tau|^2};q|\tau|) = D_{b_1,b_2+b_1}(\tau_1+1,\tau_2;q) . \tag{B.3}$$

In the detailed calculations of this paper we only need $D_{0,0}(\tau_1,\tau_2;q)$ and therefore just $\Delta_0(q)$, whose relation to the normal ordering c-function $h(t_0)$ (5.7) we will now explain.

To that end consider $F = 2h(t_0)/d$, which we write as

$$F = \sum_{n\in Z}\sqrt{n^2+q^2} \tag{B.4}$$

where $q = \omega/2t_0$. In zeta function regularisation the first step we take is to use Poisson resummation. Then

$$F = \sum_{k\in Z}\int_{-\infty}^{\infty} dy\ \mathrm{e}^{-2\pi iky}\sqrt{y^2+q^2}. \tag{B.5}$$

Rewriting the square root term as an integral,

$$\sqrt{y^2+q^2} = \frac{1}{\Gamma(-1/2)} \int_0^\infty dt\, t^{-3/2} \mathrm{e}^{-t(y^2+q^2)} \tag{B.6}$$

we can then write

$$\begin{aligned} F &= \frac{1}{\Gamma(-1/2)} \sum_{k\in Z} \int_{-\infty}^{\infty} dy\, \mathrm{e}^{-2\pi iky} \int_0^\infty dt\, t^{-3/2} \mathrm{e}^{-t(y^2+q^2)} \\ &= -\frac{2q}{\pi} \sum_{k=1}^{\infty} \frac{K_1(2\pi kq)}{k} - \frac{q^2}{2}\Gamma(-1) \\ &= 2\Delta_0(q) - \frac{q^2}{2}\Gamma(-1) \ . \end{aligned} \tag{B.7}$$

Thus we find

$$\frac{h(t_0)}{d} = \Delta_0(q) - \frac{q^2}{4}\Gamma(-1) \ . \tag{B.8}$$

Alternatively the binomial expansion of $F$ for small $q^2$ is

$$\begin{aligned} F &= q + 2\sum_{n=1}^{\infty} \sqrt{n^2+q^2} \\ &= q + 2\sum_{n=1}^{\infty}\sum_{k=0}^{\infty} \frac{\Gamma(3/2)}{\Gamma(3/2-k)} \frac{q^{2k}}{n^{2k-1}k!} \\ &= q + 2\sum_{k=0}^{\infty} \frac{\Gamma(3/2)\zeta(2k-1)}{\Gamma(3/2-k)k!} q^{2k} \\ &= q + 2\zeta(-1) + q^2\zeta(1) + 2\sum_{k=2}^{\infty} \frac{\Gamma(3/2)\zeta(2k-1)}{\Gamma(3/2-k)k!} q^{2k} \ , \end{aligned} \tag{B.9}$$

giving the expansion

$$\frac{h(t_0)}{d} = -\frac{1}{12} + \frac{q}{2} + \frac{q^2}{2}\zeta(1) + \sum_{k=2}^{\infty} \frac{(-)^k\Gamma(k-\frac{1}{2})}{\Gamma(-\frac{1}{2})\Gamma(k+1)} \zeta(2k-1) q^{2k} \ . \tag{B.10}$$

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